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THE THEOREM OF TORELLI FOR SINGULAR CURVES

BY

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ABSTRACT. Let C be a compact (singular) curve embedded in a surface. Then C carries a canonical sheaf Ω which is locally free of rank 1. Moreover, C has a generalized Jacobian J which fits in an exact sequence

$$(*) \qquad 0 \to F \to J \to A \to 0$$

of *algebraic* groups such that A is an abelian variety and $F = (\mathbf{C}^*)^r \times \mathbf{C}^s$. Let $\underline{C}$ be the set of nonsingular points of C and let θ = Zariski-closure of the image of $(\underline{C})^{(g-1)}$ in J. Then:

THEOREM. *If C is irreducible and sections of Ω map C onto X in P^{g-1} then the isomorphism class of J together with the translation class of the divisor θ on J determine the isomorphism class of X.*

As a corollary, if ψ: $C \to X$ is an isomorphism (in which case we call C nonhyperelliptic) the above data determine the isomorphism class of C. I do not know if this remains true when C is hyperelliptic.

It should be noted that the linear equivalence class of θ is not enough to determine X.

The principal idea of the proof is that of Andreotti, that is, to recover the curve as the dual of the branch locus of the Gauss map from θ to P^{g-1}; however our arguments are usually analytic.

The organization of this paper is as follows: In §1 we prove a stronger than usual version of Abel's theorem for Riemann surfaces and in §2 we extend this theorem to apply to singular curves. In succeeding sections we construct the generalized Jacobian as a complex Lie group J and embed J in an analytic fibre bundle over A with projective spaces as fibre. This we use to endow J with the structure of an algebraic group. §7 contains a miscellany of facts about branch loci and dual varieties, and in §8 the main theorems are stated and proved.

We should mention here that the variations on Abel's theorem proved in this paper (1.2.4 and 3.0.1) were proved by Severi, at least in the special case corresponding to ordinary double points [12].

1.

1.0. We are concerned here with the bilinear relations in the periods of two meromorphic 1-forms on a compact Riemann surface C of genus g. Thus let P and P' be disjoint finite subsets of C and let w and w' be smooth closed 1-forms in $C - P$ and $C - P'$ respectively. One may represent a basis $\bar{a}_1, \ldots, \bar{a}_{2g}$ of $H_1(C; Z)$ by loops a_i whose supports $|a_i|$ lie in the complement of the union of two disjoint disks B and B' through P and P'. (By a disk

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through P we shall mean a diffeomorphic image of the compact unit disk in **C** such that int $B \supset P$.) Similarly, the dual classes $\bar{a}_i^*$ in $H^1(C; Z)$ possess representatives a_i^* which are smooth closed 1-forms supported in $D = C - (B \cup B')$. The *periods* $w_i = \int_{a_i} w$ ($w_i' = \int_{a_i} w'$) do not change when each a_i is replaced by a loop homologous in $C - P$ to a_i. Stokes' theorem secures that $\int_{\partial B} w = 0$, so $w - \sum w_i a_i^*$ annihilates every element of $H_1(C - B; Z)$. As a result there exists a smooth function f (f') on C such that on a neighborhood of $C - \text{int}\, B$ ($C - \text{int}\, B'$) we have:

$$w = df + \sum w_i a_i^*, \qquad w' = df' + \sum w_i' a_i^*.$$

We let $D = C - B \cup B'$, $q_{ij} = \int_C a_i^* \wedge a_j^*$, $(w, w') = \sum w_i w_j' q_{ij}$, and note that $a_i^* | \partial D = 0$. Then application of Stokes' theorem yields:

$$\int_D w \wedge w' = \int_{\partial B} f'w - \int_{\partial B'} fw' + (w, w').$$

When w and w' are meromorphic, $w \wedge w' = 0$. Thus:

(1) $$(w, w') = \int_{\partial B'} fw' - \int_{\partial B} f'w.$$

Given a meromorphic form u defined on a neighborhood of the finite set S we let $r(u; S) = \sum\{\text{res}_p(u): p \in S\}$. Using this notation formula (1) becomes

(2) $$(w, w') = 2\pi\sqrt{-1}\,(r(fw'; P') - r(f'w; P)).$$

1.1. We specialize to the case in which w' has only simple poles and integral residues. If $\text{res}_p w' = n(p)$, then $r(fw'; P') = \sum\{n(p)f(p): p \in P'\}$. Choosing a 1-chain c' in B' such that $\partial c' = d' = \sum\{n(p)p: p \in P'\}$ we have:

(1) $$(w, w') = 2\pi\sqrt{-1}\left(\int_{c'} w - r(f'w; P)\right).$$

The group $H_1(C - P; Z)$ is generated by the cycles a_i together with the boundaries b_p of disjoint disks $D_p \subset \text{int}\, B$ centered at $p \in P$. After fixing a choice of such disks, for any chain c in $C - P$ such that $\partial c = d'$ there exist integers l_i and m_p such that $c' - c$ is homologous in $C - P$ to $a + b$, wherein $a = \sum l_i a_i$ and $b = \sum m_p b_p$. Then:

(2) $$(w, w')/2\pi\sqrt{-1} + r(f'w; P) = \int_{a+b+c} w.$$

Leaving the chains a, b, and c fixed, (2) remains valid when w and w' are replaced by $w + h$ and $w' + h'$, provided that h' be holomorphic and that h be meromorphic with poles in P only.

1.2. The space of holomorphic 1-forms on C will be denoted by H. More generally, given a divisor d, the symbol $H(d)$ shall stand for the space of meromorphic forms w whose divisors (w) satisfy $(w) + d > 0$. Finally H_P shall denote the space of meromorphic forms with poles in the set P. If we arrange the elements h_i of a basis for H in a column vector $\underline{h}$ of length g and the cycles a_i in a row vector $\underline{a}$, then the $g \times 2g$ period matrix $\Omega = \int_{\underline{a}} \underline{h}$ satisfies

(1) $\Omega Q' \Omega = 0$ and

(2) $\sqrt{-1}\, \Omega Q' \bar{\Omega}$ is positive definite hermitian ($Q = (q_{ij})$).

The first property follows immediately from (1.0.1) taking w and w' in H while the second follows from the fact that $\sqrt{-1}\, \Omega Q' \bar{\Omega}$ is the matrix relative to the basis $\underline{h}$ of the hermitian positive definite form $\langle w, w' \rangle = \sqrt{-1} \int_C w \wedge \bar{w}'$ on H. As a consequence of (1) and (2) Ω has rank $2g$ over R. Accordingly, a meromorphic form w' may be normalized by adding an appropriate holomorphic form so that $\underline{w}' = \int_{\underline{a}} w' = 2\pi\sqrt{-1}\, \underline{r}$ where $\underline{r} \in R^{2g}$. Then $(w, w')/2\pi\sqrt{-1} = \underline{w} Q' \underline{r}$ is a real linear combination of the periods of w.

(3) LEMMA. *Let w' be a normalized meromorphic form with simple poles and integral residues and let c be an integral* 1*-chain which avoids the finite set P and which satisfies $\partial c = d' = \Sigma \mathrm{res}_p(w')p$. Choose integral cycles a and b as in* (1.1) *so that* (1.1.2) *holds for all w in H_P. If $\int_c w = 0$ for all $w \in H$ then*

(a) $\int_{b+c} w = r(f'w; P)$ *for all w in H_P and*

(b) $\underline{w}'/2\pi\sqrt{-1}$ *is in Z^{2g}.*

PROOF. If $a = \underline{a}({}^t\underline{l})$ and $\underline{r} = \underline{w}'/2\pi\sqrt{-1}$ then

$$\int_a w - (w, w')/2\pi\sqrt{-1} = \underline{w}({}^t\underline{l} - Q\,{}^t\underline{r})$$

and the vector $\underline{m} = {}^t\underline{l} - Q\,{}^t\underline{r}$ is real. Furthermore, $\overline{\underline{w}\underline{m}} = 0$ for all w in H. Hence $\Omega \underline{m} = 0$ and since Ω has rank $2g$, $\underline{m} = 0$. This proves (a), and (b) follows from the fact that Q is unimodular.

(4) THEOREM. *Let d' be a divisor on C whose support avoids the finite set P. Given a subset F of H_P such that $H \subset F$ the following are equivalent:*

(A) *There exist a meromorphic function g on C such that $(g) = d'$ and a branch f of* $\log(g)$ *defined in a neighborhood of P such that $r(fw; P) = 0$ for all $w \in F$.*

(B) *There exists an integral chain c in $C - P$ such that $\partial c = d'$ and $\int_c w = 0$ for all w in F.*

PROOF. Assuming (A), let $w' = d\log(g)$ and choose a chain c supported in $C - P$ such that $\partial c = d'$, a continuous branch f' of $\log(g)$ in a disk through P, and cycles a and b as in (1.1) so that (1.1.2) holds for all w in H_P. By hypothesis (g is single-valued) the vector $\underline{r} = {}^t\underline{w}'/2\pi\sqrt{-1}$ is in Z^{2g} and $\int_{a+b+c} w = r(f'w; P) + \underline{w} Q \underline{r}$ for all w in H_P. If $f = f' + 2\pi\sqrt{-1}\, m_p$ near p, let $b' = b + \Sigma m_p b_p$, $a' = a - \underline{a} Q \underline{r}$, and $c' = a' + b' + c$. Then $\int_{c'} w = r(fw; P) = 0$ for all w in F.

Assuming (B), we have that $d' = \partial c$ so degree$(d') = 0$. Hence we may construct a normalized meromorphic 1-form w' such that $\Sigma \text{res}_p(w')p = d'$ and, by Lemma (3), $\underline{w}'/2\pi\sqrt{-1} \in Z^{2g}$. Thus there exists a meromorphic function g such that $w' = d\log(g)$ and choosing a continuous branch f' of $\log(g)$ on a disk through P we have $\int_b w = r(f'w; P)$ for all w in f (also by Lemma (3)). If $b = \Sigma m_p b_p$ we let $f = f' - 2\pi\sqrt{-1}\, m_p$ near p and the theorem is proved.

1.3. Given an effective divisor d whose support P avoids the support of d' we let $F = H(d)$ and apply the above theorem.

EXAMPLE 1. $d = p_1 + p_2, p_1 \neq p_2$. Then it follows that there exists a chain c such that $\partial c = d'$ and $\int_c w = 0$ for all w in $H(d)$ iff there is a meromorphic function g such that $(g) = d'$ and $g(p_1) = g(p_2)$.

EXAMPLE 2. $d = 2p$. In this case $\partial c = d'$ and $\int_c w = 0$ for all w in $H(d)$ iff there exists g meromorphic on C such that $(g) = d'$ and $dg|_p = 0$.

2.

2.0. Let C be a curve (reduced, compact, complex 1-dimensional analytic subvariety) on a complex analytic surface V (complex 2-dimensional manifold), and let S be the singular locus of C. There are a normalization C' and a surjective canonical map $\pi: C' \to C$ such that π maps $\underline{C}' = C' - \pi^{-1}(S)$ isomorphically onto $\underline{C} = C - S$.

Following Kodaira, for each p in C we choose coordinates (x, y) and a function R on a neighborhood U of P and require that $x(p) = y(p) = 0$, $C \cap U = \{q: R(q) = 0\}$, and that when p is simple $R(x, y) = y$ and when p is singular $R(x, y) = y^m + a_1(x)y^{m-1} + \cdots + a_m(x)$ where $a_i(0) = 0$ and $a_i(x)$ is analytic. Then the meromorphic form $\sigma = dx/(\partial R/\partial y)$ is defined near p and $\pi^*\sigma$ has poles only when p is singular. When p is singular and $\pi(q) = p$ then $\pi^*\sigma$ has a pole of positive order c_q at q. The positive divisor $c = \Sigma\{c_q q: \pi(q) \in S\}$ has an associated line bundle $[c]$ called the conductor of C in C'.

2.1. The *canonical bundle* K of C is by definition the restriction to C of the bundle $K_V \otimes [C]$ where K_V is the canonical bundle of V and $[C]$ is the line bundle of the curve C considered as an effective divisor on V.

2.2. One verifies that in the intersection of two neighborhoods U_a and U_b with defining equations R_a and R_b as specified in (2.0) the ratio $(\sigma_a/\sigma_b)|C$ is holomorphic without zeros and coincides with the ratio of nonvanishing sections of K in U_a and U_b respectively. Accordingly, there is no harm in regarding σ_a as a nonvanishing section of K on $U_a \cap C$. We shall denote by H the space of holomorphic sections of K.

2.3. Given a meromorphic section w of K on an open set W there are functions f_a on $U_a \cap W$ such that $w = f_a\sigma_a$. Thus we may define $\pi^*w = f_a \circ \pi(\pi^*\sigma_a)$, and π^*w is a meromorphic differential form on $\pi^{-1}(W)$. Similarly, given a 1-chain c in $C - S$ which avoids the pole set of w we let

$\int_c w = \int_{\pi^{-1}(c)} \pi^* w$ and given a meromorphic section w of K over a neighborhood of a set P we define $r(w; P) = r(\pi^* w; \pi^{-1}(P))$.

2.4. The bundle K lifts to a bundle $K \circ \pi$ on C'. Given a fixed section s of $[c]$, vanishing on c, one may define an isomorphism L from the sheaf $\mathcal{O}(K \circ \pi)$ of holomorphic sections of $K \circ \pi$ to the sheaf $\mathcal{O}(k \otimes [c])$ (where k is the canonical bundle of C') as follows: If w is a section of $K \circ \pi$ over W then $w = f_a \sigma_a$ on $W \cap \pi^{-1}(U_a)$ and we let $L(w) = f_a(\pi^* \sigma_a) \otimes s$. Thus $K \circ \pi$ is linearly equivalent to $k \otimes [c]$.

2.5. Let $\mathcal{O}$ (respectively $\mathcal{O}'$) denote the sheaf of germs of holomorphic functions on C (C') and let M (M') denote the total quotient sheaf of $\mathcal{O}$ ($\mathcal{O}'$). The natural homomorphism π^*: $M_p \to (\pi_* M')_p$ ($\pi_* M'$ is the direct image of M') is an isomorphism while the restricted map π^*: $\mathcal{O}_p \to (\pi_* \mathcal{O}')_p$ is not in general surjective. However, the image of $\pi^*|\mathcal{O}_p$ may be characterized by residues as follows:

THEOREM 1. *Given f' in $\pi_*(\mathcal{O}')_p$ there exists f in $\mathcal{O}_p$ such that $f \circ \pi = \pi^* f = f'$ iff $r(f' g \pi^* \sigma_a; \pi^{-1}(p)) = 0$ for all g in $\mathcal{O}_p$ (cf.* [3]).

COROLLARY 2. *If f' is holomorphic in $\pi^{-1}(U)$ and vanishes to order c_q at each q in $\pi^{-1}(S \cap U)$ then there exists f holomorphic on U such that $f' = \pi^* f$.*

COROLLARY 3. *If $w \in \mathcal{O}(K)_p$ then $r(\pi^* w; \pi^{-1}(p)) = 0$.*

PROOF. The constant 1 is holomorphic on C.

COROLLARY 4. *If w' is a meromorphic form on a neighborhood of $\pi^{-1}(p)$ then $w' = \pi^* w$ for some $w \in \mathcal{O}(K)_p$ iff $r((\pi^* f) w'; \pi^{-1}(p)) = 0$ for all f in $\mathcal{O}_p$.*

COROLLARY 5. *Let U be an open set in C and let w' be a holomorphic differential form in $\pi^{-1}(U)$. Then there exists a holomorphic section w of K over U such that $\pi^* w = w'$.*

PROOF. In $\pi^{-1}(U \cap U_a)$ we have $w' = f'_a \pi^* \sigma_a$ and since w' is holomorphic and $\pi^* \sigma_a$ has poles of order c_q at q in $\pi^{-1}(p)$ for each p in $S \cap U$ the function f'_a must vanish to order c_q at q. Hence there exists a holomorphic function f_a on $U \cap U_a$ such that $f'_a = \pi^* f_a$ and letting $w = f_a \sigma_a$ on $U \cap U_a$ we have the desired form.

LEMMA 6. *Let w be a section of K over an open set $U \subset C$ such that for each irreducible component X of C, $r(\pi^* w; \pi^{-1}(U \cap S) \cap X') = 0$. Then there exists a section v in H such that $\pi^* v - \pi^* w$ is holomorphic on $\pi^{-1}(U)$. (X' is the normalization of X and is canonically contained in C'.)*

PROOF. The residue condition insures that on each X' there is a meromorphic form $v_{X'}$ such that $v_{X'}$ is holomorphic outside $\pi^{-1}(U)$ and $v_{X'} - \pi^* w$ is holomorphic on $\pi^{-1}(U) \cap X'$. Let $v'|X' = v_{X'}$. Then for each p

in $S \cap U$ and f in $\mathcal{O}_p$,

$$r\big((\pi^*f)v'; \pi^{-1}(p)\big) = r\big(\pi^*(fw); \pi^{-1}(p)\big) = 0.$$

Hence by Corollary 4 there exists v in H such that $\pi^*v = v'$.

COROLLARY 7. *If C is irreducible and $w \in \mathcal{O}(K)_p$ then there exists v in H such that $\pi^*(v - w)$ is holomorphic near p and π^*v is holomorphic outside $\pi^{-1}(p)$.*

COROLLARY 8. *If w' is a meromorphic form on $\pi^{-1}(U)$ with simple poles all of which lie on $\pi^{-1}(S \cap U)$ and if $r(w'; \pi^{-1}(p)) = 0$ for all p in $S \cap U$ then there exists a holomorphic section w of K on U such that $\pi^*w = w'$.*

PROOF. For all f in $\mathcal{O}_p$ we have

$$r\big(\pi^*fw'; \pi^{-1}(p)\big) = r\big((\pi^*f - f(p))w'; \pi^{-1}(p)\big) + f(p)r\big(w'; \pi^{-1}(p)\big).$$

The first term vanishes because the form $(\pi^*f - f(p))w'$ is holomorphic near $\pi^{-1}(p)$ and the second term vanishes by the residue condition on w'.

LEMMA 9. *If $w \in \mathcal{O}(K)(U)$ then there exist $u \in \mathcal{O}(K)(U)$ and $v \in H$ such that π^*u has only simple poles, π^*v has zero residues and $w = u + v$.*

PROOF. For each component X of C either $X \subset U$ or $X - U$ is not empty. In the first case $r(\pi^*w; X') = 0$. Thus it is possible to construct a form $t_{X'}$ on X' with simple poles such that $\pi^*w - t_{X'}$ has zero residues everywhere on X'. In the second case, by allowing $t_{X'}$ to have poles on $X' - \pi^{-1}(U)$, we may construct $t_{X'}$ having simple poles in $X' \cap \pi^{-1}(U)$ and such that $\pi^*w - t_{X'}$ has zero residues in $X' \cap \pi^{-1}(U)$. Let $t'|X' = t_{X'}$. From Corollary 8 we conclude there exists t in $\mathcal{O}(K)(U)$ such that $\pi^*t = t'$. Then $\pi^*(w - t)$ has zero residues in $\pi^{-1}(U) \cap X'$ and by Lemma 6 there exists v in H such that $\pi^*(w - t - v)$ is holomorphic on U. Thus $u = (w - t - v) + t = w - v$ and v are the desired forms.

2.6. Let s be a simple closed curve in C which crosses the singular set S. Then s determines a finite number of paths s_i' in C' (each parameterized by the unit interval) such that $s = \pi s_1' \pi s_2' \cdots \pi s_m'$, and such that the end points of each s_i' lie in $S' = \pi^{-1}(S)$ and no other point of s_i' is in S'. We shall call s admissible if $s_i'(1) \neq s_{i+1}'(0)$ for all i (consider the paths s_i' to be indexed by the integers mod m). Then the following lemma is a clear consequence of (2.5.8).

LEMMA 1. *If s is an admissible closed curve on C then there exists w in H such that π^*w has simple poles exactly at the points $s_i'(e)$ ($e = 0$ or 1) with residue $(-1)^e$ at $s_i'(e)$.*

2.7. A function $\sigma: \pi^{-1}(S) \to \mathbf{C}$ will be called a *residue* cycle iff

(a) $\Sigma_{\pi(p)=q}\sigma(p) = 0$ for all $q \in S$;

(b) $\Sigma_{p\in X}\sigma(p) = 0$ for each component X of C'.

A section $w \in H$ is *associated* with the residue cycle σ iff

(a) π^*w has at most simple poles (i.e. $\text{ord}_p\pi^*w \geq -1$).

(b) $\text{res}_p\, \pi^*w = \sigma(p)$ for all p.

Finally, we call $w \in H$ *simple* iff w is associated to a residue cycle σ such that $\sigma(p) = \pm 1$ for all $p \in |\sigma|$. An inductive argument suffices to show that the simple sections span the space $H_1 = \{w\colon \text{ord}_p\, \pi^*w \geq -1 \text{ for all } p\}$.

3.

3.0. In this section we extend Theorem 1.2.4 to singular curves. Given a finite set $P \subset C$ we let H_P be the vector space of meromorphic sections of K which are holomorphic in $C - P$.

THEOREM 1. *Let d be a Cartier divisor on C whose support avoids the finite set P and the singular locus S. Given a subset F of H_P such that $H \subset F$, the following are equivalent*:

(A) *There exist a meromorphic function g on C such that $(g) = d$ and a branch f of* $\log(g)$ *in a neighborhood of P such that $r(fw; P) = 0$ for all w in F.*

(B) *There exists an integral chain γ in $C - E$ ($E = P \cup S$) such that $\partial\gamma = d$ and $\int_\gamma w = 0$ for all w in F.*

PROOF. Let $E' = \pi^{-1}(E)$, $F' = \{\pi^*w\colon w \text{ is in } F\}$, and d' be the unique 0-chain such that $\pi_*(d') = d$. Note that if a branch f of $\log(g)$ satisfies $r(fw; P) = 0$ for all w in F and if g is holomorphic on $E = P \cup S$ then we may extend f to a neighborhood of E and have $r(fw; E) = 0$ for all w in F (cf. 2.5.3). Conversely, given a branch f of $\log(g)$ defined in a neighborhood of E then $r(fw; p) = 0$ for all w in F and all p in $S - P$ so $r(fw; E) = r(fw; P)$.

Assume (A) and apply 1.2.4 to the functions $f' = f$ and $g' = g$, divisor d', and sets F' and E' to construct a chain γ' such that $\partial\gamma' = d'$ and $\int_{\gamma'} w = 0$ for all w in F'. Letting $\gamma = \pi_*\gamma'$ we have $\int_\gamma w = \int_{\gamma'}\pi^*w = 0$ for all w in F. Thus (A) implies (B).

Assume (B), let $\pi_*\gamma' = \gamma$ and proceed by induction on the number of irreducible components of C.

If C is irreducible apply 1.2.4 to the set F' and chain γ' to produce a meromorphic function g' on C' and a branch f' of $\log(g')$ defined near E' such that $r(f'w; E') = 0$ for all w in F'. If $u \in \mathcal{O}(K)_p$ then there exists w in H such that $\pi^*(u - w)$ is holomorphic near $\pi^{-1}(p)$ and π^*w is holomorphic outside $\pi^{-1}(p)$. Then $r(f'\pi^*u; \pi^{-1}(p)) = r(f'\pi^*w; \pi^{-1}(E')) = 0$ so there exists f in $\mathcal{O}_p$ such that $f' = f \circ \pi$ near p. Then $g = e^f$ is holomorphic and invertible near p and $g' = g$. Moreover $r(fw; P) = 0$ for all w in F as required.

If $C = C_1 \cup C_2$, C_1 irreducible and $C_1 \cap C_2$ finite, then let K_i be the canonical bundle of C_i, $H_i = H^0(C_i, \mathcal{O}(K_i))$. Choose a section s_i of $[C_i]$ vanishing exactly on C_i and define a map T_i: $H_i \to H \subset F$ by $T_i(w) = ws$ (extended by zero over C). If γ is our chain on C and γ_i its restriction to C_i then $\int_{\gamma_i} w = \int_\gamma T_i(w) = 0$ for all w in H_i. Now use the induction hypothesis on the chains γ_i and sets $F_i = H_i$ to produce meromorphic functions g_i on C_i such that $(g_i) = d_i = d|C_i$. Let $g'|C_i' = g_i$ and select a branch f' of $\log(g')$ defined near E' and a cycle b on C' (as in 1.1) so that $\int_{b+c} w = r(f'w; E')$ for all w in $H_{E'}$. We shall require that f' be chosen so that $f'|C_i' \cap \pi^{-1}(p)$ is constant for each p in the complement of the support of the divisor d_i.

Lemma 2. *If X is a connected component of C_2 and $\{p, q\} \subset X \cap C_1$ then $g_1(p)/g_1(q) = g_2(p)/g_2(q)$.*

Proof. Let s_1 be a simple path from p to q in C_1 which avoids $S - \{p, q\}$ and let t be a simple path from q to p in C_2 such that $s = s_1 t$ is an admissible loop in C. Then s lifts to a collection of disjoint paths s_j' ($j = 1, \ldots, m$) in C' such that $s_1'(0) = p = s_m'(1)$ and $s_1'(1) = q = s_2'(0)$. Let w be the section of K constructed as in 2.6.1 such that $\pi^* w$ has a simple pole and residue $(-1)^e$ at $s_j'(e)$ ($e = 0$ or 1). Then:

$$(**) \qquad \int_b \pi^* w = r(f'\pi^* w; E') = \sum_j f'(s_j'(0)) - f'(s_j'(1)).$$

Now $\int_b \pi^* w = 2\pi\sqrt{-1}\, m$ for some integer m so applying the exponential function to both sides of (**) we have:

$$1 = g'(s_j'(0))/g'(s_j'(1)).$$

Since $g'(s_j'(1)) = g_2(s_j'(1)) = g_2(s_{j+1}'(0)) = g'(s_{j+1}'(0))$ if $2 \leqslant j \leqslant m - 1$ the desired result follows.

We choose p_X in $X \cap C_1$ for each connected component X of C_2 such that $X \cap C_1 \neq \varnothing$ and require that $g_2(p_X) = g_1(p_X)$ (multiplying g_2 on X by a constant if necessary). Then for all p in $C_1 \cap C_2$ we have $g_1(p) = g_2(p)$ by Lemma 2. It follows that if $w \in \mathcal{O}(K)_p$ (p not in the support of d) and $\pi^* w$ has poles of order 1 at most then $r(f'\pi^* w; \pi^{-1}(p)) = f(p) r(\pi^* w; \pi^{-1}(p))$ and the left side of this equation is zero. For arbitrary w in $\mathcal{O}(K)_p$ there exist u and v as in (2.5.6) such that $w = u + v$ near p. Thus

$$r(f'\pi^* w; \pi^{-1}(p)) = r(f'\pi^* u; \pi^{-1}(p)) + r(f'\pi^* v; \pi^{-1}(p))$$
$$= r(f'\pi^* v; E') = \int_{b+c} \pi^* v = 0$$

since v is in F and $\pi^* v$ has null residues. By 2.5.1 there exists f in $\mathcal{O}_p$ such that $f' = f \circ \pi$. We let $g|C_1 = g_1$, $g|C_2 = g_2$ and note that $g = e^f$ in a neighborhood of p. Hence g is holomorphic and invertible in a neighborhood of every point not in the support of d. Furthermore, $r(fw; P) = r(f'\pi^* w; P') =$

0 for all w in F. This proves (B) implies (A).

4.

4.0. *The generalized Jacobian.* We let $\mathcal{O}^*$ and M^* denote respectively the subsheaf of units in $\mathcal{O}$ and the sheaf of germs of meromorphic functions which do not vanish on any component of C. Then $\mathfrak{D} = M^*/\mathcal{O}^*$ is the sheaf of germs of Cartier-divisors and $D = H^0(C, \mathfrak{D})$ is the group of Cartier divisors on C. If $d \in D$ then $|d| = \{p\colon d \text{ is not trivial at } p\}$. We let $\underline{C} = C - S$ ($\underline{C}' = C' - \pi^{-1}(S)$) and $\underline{D} = \{d \in D\colon |d| \subset \underline{C}\}$. The exact sequence

$$1 \to \mathcal{O}^* \to M^* \to \mathfrak{D} \to 0$$

leads to the sequence

$$D \to H^1(C, \mathcal{O}^*) \to H^1(C, M^*) \to 0.$$

It has been noted that M^* is naturally isomorphic to $\pi_*(M')^*$. Hence we have: $H^1(C, M^*) \cong H^1(C, \pi_*(M')^*) \cong 0$. Thus the homomorphism sending the divisor d to its associated $\mathbf{C}^*$ bundle $[d]$ in $H^1(C, \mathcal{O}^*)$ is surjective.

4.1. The exact sequence $0 \to Z \to \mathcal{O} \xrightarrow{\exp(2\pi i\cdot)} \mathcal{O}^* \to 1$ yields a homomorphism ch: $H^1(C, \mathcal{O}^*) \to H^2(C, Z)$. Let $G = \{[d]\colon d \in \underline{D} \text{ and } \mathrm{ch}[d] = 0\}$ and $G_0 = \{d\colon d \in \underline{D} \text{ and } [d] \text{ is trivial}\}$. Then the *generalized Jacobian* of C is the group $J = G/G_0$. Giving a divisor in G is equivalent to giving an integral zero chain $\Sigma n_p p$ on $\underline{C}$ such that $\Sigma\{n_p\colon p \in X\} = 0$ for each irreducible component X of C. Let $S_1(\underline{C}, Z)$ be the group of integral 1-chains in $\underline{C}$ and define $S_1(\underline{C}, Z) \hat{\to} H^*$ by $(\hat{l})(w) = \int_l w$. By Theorem 3.0.1, $\hat{l} = 0 \Leftrightarrow \partial l \in G_0$. Now let Γ be the image of the induced homomorphism $H_1(\underline{C}, Z) \to H^*$. Given $d \in G$ we choose $l \in S_1(\underline{C}', Z)$ such that $\partial l = d$ and let $f(d)$ be the image in H^*/Γ of l. The above remarks show that f induces an injective homomorphism (hereafter denoted by f) from J to H^*/Γ.

4.2. Let $H' = H^0(C', \Omega')$. By (2.5.5) we may regard H' as a subspace of H. Let $H'' = H/H'$ and define maps and spaces by the following exact commutative diagrams:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma'' & \xrightarrow{\lambda''} & (H'')^* & \xrightarrow{p''} & F & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i_p} & & \downarrow{\scriptstyle i_H} & & \downarrow{\scriptstyle i_F} & & \\
0 & \longrightarrow & \Gamma & \xrightarrow{\lambda} & H^* & \xrightarrow{p} & H^*/\Gamma & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \rho_\Gamma} & & \downarrow{\scriptstyle \rho_H} & & \downarrow{\scriptstyle \rho_J} & & \\
0 & \longrightarrow & \Gamma' & \xrightarrow{\lambda'} & (H')^* & \xrightarrow{p'} & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & V & \longrightarrow & V & \longrightarrow & 0 & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T & \longrightarrow & H_1(\underline{C}, Z) & \xrightarrow{\pi^{-1}} & H_1(C', Z) & \longrightarrow & 0\\
 & & {\scriptstyle \Lambda''}\downarrow & & \downarrow{\scriptstyle \Lambda} & & \downarrow{\scriptstyle \Lambda'} & & \\
0 & \longrightarrow & \Gamma'' & \xrightarrow{i_\Gamma} & \Gamma & \xrightarrow{\rho_\Gamma} & \Gamma' & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Let $n = \operatorname{card} \pi^{-1}(S)$ and r be the number of components of C. The module T is generated by circles around points in $\pi^{-1}(S)$ and has rank $n - r$. That Γ' is a discrete subgroup of $(H')^*$ of rank $2g'$ over R follows from (1.2). (We let $g' = \dim H'$ and $g = \dim H$.)

4.3. In H'' we have canonical subspaces $E_1 = \operatorname{image}\{w\colon \operatorname{ord}_p \pi^*w > -1$ for all $p\}$ and $E_2 = \operatorname{image}\{w\colon \operatorname{res}_p \pi^*w = 0$ for all $p\}$. By Lemma 2.5.9, $H'' = E_1 + E_2$. Now the elements of T annihilate E_2 so $\Gamma'' \subset E_2^{\perp} = E_1^*$. By (2.7) we may choose a basis $\eta_1, \ldots, \eta_s$ of E_1 coming from simple sections of H. If $t \in T$ then $t \in V \Leftrightarrow t(\eta_i) = 0$ for all i. Thus we have:

$$\operatorname{rank} V = \operatorname{rank} T - s \quad \text{and} \quad \operatorname{rank} \Gamma'' = s \; (= \dim E_1).$$

(1) Lemma. (a) *Γ is a discrete subgroup of H^* of rank $2g' + s$.*

(b) *The natural map $\varphi\colon \Gamma'' \otimes_Z \mathbf{C} \to E_1^*$ is an isomorphism.*

Proof. (a) If $\gamma_n \in \Gamma$ and $\gamma_n \to 0$ then $\rho_\Gamma(\gamma_n) \to 0$ in Γ' and since Γ' is discrete this means $\gamma_n \in \Gamma''$ for n large. Then $\gamma_n(\eta_i) \to 0$ as $n \to \infty$ and since $\gamma_n(\eta_i) \in Z$ this implies $\gamma_n(\eta_i) = 0$ for large n. This being true for all i we have shown $\gamma_n = 0$ for large n and (a) is proved.

(b) Since the two vector spaces in question have the same dimension it suffices to show $(\operatorname{im} \varphi)^{\perp} = 0$. If $w \in H$ is such that the image of w in E_1, $\bar{w}$, is also in $(\operatorname{im} \varphi)^{\perp}$ then for every circle t about a point p in $\pi^{-1}(S)$ we have $\int_t \pi^*w = 0$. Thus $\operatorname{res}_p w = 0$ for all $p \in \pi^{-1}(S)$; hence $\bar{w} \in E_1 \cap E_2 = \{0\}$; and we have shown $\operatorname{im} \varphi = E_1^*$.

4.4. A consequence of this lemma is that H^*/Γ is a complex Lie group. Choose a nonsingular base point b_i on each irreducible component C_i of C. We define the map $u\colon \underline{C} \to J$ by:

$$u(p) = [p - b_i] \quad \text{if } p \in C_i.$$

One verifies that $f \circ u$ is holomorphic. Moreover, regarding the cotangent bundle $T^*(H^*/\Gamma)$ as a trivial bundle with fibre H we have: $(f \circ u)^*w = w$.

4.5. Denoting the s-fold symmetric power of $\underline{C}$ by $\underline{C}^{(s)}$ we extend the map u to $u_s\colon \underline{C}^{(s)} \to J$ by addition. A familiar calculation [2] shows that:

$$(1) \quad \begin{aligned} \big((f \circ u_s)_* T_d(\underline{C}^{(s)})\big)^{\perp} &= \{w \in H\colon w \text{ vanishes on the divisor } d\} \\ &= H^0(C; \Omega(-d)) \qquad \big(\Omega = \mathcal{O}(K) \text{ and } \Omega(-d) = \mathcal{O}\big(K \otimes [d]^{-1}\big)\big). \end{aligned}$$

It is easy to choose the points $p_1, \ldots, p_g$ in $\underline{C}$ such that if $d = p_1 + \cdots + p_g$ then $H^0(C; \Omega(-d)) = 0$. At such a point $f \circ u_g$ has rank g, and it follows that the subgroup $f(J)$ contains an open subset of H^*/Γ. Since H^*/Γ is connected, this proves:

(2) Lemma. *$f\colon J \to H^*/\Gamma$ is an isomorphism.*

We therefore have an exact sequence $0 \to F \to J \to A \to 0$ in which A is an abelian variety (the Jacobian of C') and $F = (H'')^*\Gamma'' \cong (E_1^*/\Gamma'') + E_2^*$.

(3) Theorem. *The group J is a complex analytic principal bundle over A with fibre F. Furthermore this bundle is flat in the sense that the transition functions are constant relative to a suitable local trivialization.*

Proof. Let $\{U_\alpha\}$ be an open cover of A by convex sets evenly covered by the covering map $p'\colon (H')^* \to A$ and choose a fixed splitting $\Delta\colon H \to H'$ of the sequence $0 \to H' \to H \to H'' \to 0$. Make a choice p_α of $(p')^{-1}$ on U_α and let $s_\alpha(x) = p\Delta^* p_\alpha(x)$. Then $\rho_J(s_\alpha(x)) = x$ and $(s_\alpha - s_\beta)(x) = p\Delta^*(p_\alpha(x) - p_\beta(x)) = p\Delta^*(\lambda_{\alpha\beta}) = f_{\alpha\beta}$ where $\lambda_{\alpha\beta} \in \Gamma'$ is independent of x. Then define φ_α: $U_\alpha \times F \to \rho_J^{-1}(U_\alpha)$ by $\varphi_\alpha(x, f) = s_\alpha(x) + f$ and note that φ_α is an isomorphism and $\varphi_\beta^{-1}\varphi_\alpha(x, f) = (x, f + f_{\alpha\beta})$.

4.6. In order to exhibit the structure of F we choose a basis $\gamma_1, \ldots, \gamma_s$ for Γ'', the dual basis $\eta_1, \ldots, \eta_s$ for E_1, and another basis $\zeta_1, \ldots, \zeta_d$ for E_2. Given $x \in F$ we let $\varphi(x) = (e^{x(\eta_1)}, \ldots, e^{x(\eta_s)}, x(\zeta_1), \ldots, x(\zeta_d))$. Then φ: $F \to E(s, d) = (\mathbf{C}^*)^s \times \mathbf{C}^d$ is an isomorphism and gives F the structure of an algebraic group. Another choice of bases yields a different isomorphism φ_1, but then there is an algebraic isomorphism $\Psi\colon E(s, d) \to E(s, d)$ such that $\Psi \circ \varphi = \varphi_1$.

5.

5.0. We define a representation $\lambda\colon E(s, d) \to \mathrm{Gl}(s + d + 1, \mathbf{C})$ *via*

$$\lambda(x_1, \ldots x_s, y_1, \ldots y_d) = \begin{pmatrix} x_1 & & 0 & & & & \\ & \ddots & & & 0 & & \\ 0 & & x_s & & & & \\ & & & 1 & & 0 & y_1 \\ & 0 & & & 1 & & \vdots \\ & & & & & \ddots & y_d \\ & & & 0 & & & 1 \end{pmatrix}.$$

This yields a vector bundle E of rank $t = s + d + 1$ associated to the principal bundle J, and we let $\tilde{J}$ be the fibre bundle associated to the action of F as projective linear transformations on the space P of lines in $\mathbf{C}^t$. By a theorem of Kodaira [4], $\tilde{J}$ is a projective algebraic manifold.

We note that E (hence $\tilde{J}$) is topologically trivial over A since the transition matrices for E indicate that E contains $s + d$ linearly disjoint flat line bundles L_i such that $E/\bigoplus_i L_i$ is trivial, and flat line bundles are topologically trivial.

5.1. Accompanying the representation λ is an obvious embedding ε: $F \to P$ such that:

(a) $\lambda(f)\varepsilon(g) = \varepsilon(fg)$,

(b) $\varepsilon(F)$ is Zariski-open in P,

(c) $\varepsilon(x) = [\varphi(x), 1]$.

This allows us to construct an embedding μ: $J \to \tilde{J}$ so that the following diagram commutes:

$$\begin{array}{ccccc} F & \xrightarrow{i_F} & J & \xrightarrow{\rho_J} & A \\ {\scriptstyle\varepsilon}\downarrow & & {\scriptstyle\mu}\downarrow & & {\scriptstyle\mathrm{id}}\downarrow \\ P & \xrightarrow[i_P]{} & \tilde{J} & \xrightarrow[\tilde{\rho}]{} & A \end{array}$$

5.2. If we choose an embedding ε_1: $F \to P$ and action λ_1 corresponding to a different isomorphism φ_1: $F \to E(s, d)$ then:

(1) Lemma. *There exists a birational correspondence* $\tilde{\Lambda} \subset \tilde{J} \times \tilde{J}_1$ *such that*

$$\begin{array}{ccccc} & & \tilde{\Lambda} & & \\ & \swarrow & & \searrow & \\ \tilde{J} & & & & \tilde{J}_1 \\ & \searrow & & \swarrow & \\ & & A & & \end{array}$$

commutes and which induces an algebraic isomorphism $\mu(J) \cong \mu_1(J)$.

Proof. We let Λ_P be the closure in $P \times P$ of the correspondence $\{(\varepsilon(f), \varepsilon_1(f)): f \in F\}$, and let F act on $P \times P$ by $f \cdot (x, y) = (\lambda(f)x, \lambda_1(f)y)$. Then $F \cdot \Lambda_P \subset \Lambda_P$. By a theorem of Borel the bundle associated to the bundle J, with fibre $P \times P$, is projective algebraic and contains the bundle with fibre Λ_P as a subvariety.

5.3. The graph $\mathfrak{A}_F$ of addition in F is an analytic subset of F^3. If we let $t = s + d + 1$ and let $x = [x_1, \ldots, x_t]$ denote the homogeneous coordinates of a point in the projective space P then:

$$(x, y, z) \in \mathfrak{A}_F \Leftrightarrow (*) \begin{array}{ll} z_t x_i y_i = x_i y_t z_i, & 1 \leqslant i \leqslant s, \\ z_t(x_i y_t + x_t y_i) = z_i x_t y_t, & s + 1 \leqslant i \leqslant s + d. \end{array}$$

We let $V \subset P^3$ be the subvariety defined by the system $(*)$ and verify that:

$$V \cap (F \times F \times P) = V \cap (F \times P \times F) = V \cap (P \times F \times F) = \mathfrak{A}_F.$$

Thus the Zariski-closure $\mathfrak{A}_P$ of $\mathfrak{A}_F$ is an irreducible algebraic subvariety of P^3 in which $\mathfrak{A}_F$ is a Zariski-open set.

5.4. Now let $\mathfrak{A}_A$ and $\mathfrak{A}_J$ be the graphs in A^3 and J^3 of addition on A and J respectively. It is clear that $\mathfrak{A}_J$ is a fibre bundle over $\mathfrak{A}_A$ with fibre $\mathfrak{A}_F$. The (usual) closure of $\mathfrak{A}_J$ in $\tilde{J}$ is then a fibre bundle $\mathfrak{A}$ over $\mathfrak{A}_A$ with fibre $\mathfrak{A}_P$. Hence $\mathfrak{A}$ is algebraic and $\mathfrak{A}_J = \mathfrak{A} \cap J^3$ is a Zariski-open dense set in $\mathfrak{A}$. This shows that J is an algebraic group with the algebraic structure induced from the embedding $J \to \tilde{J}$. While the compactification $\tilde{J}$ is not canonical the induced structure is by (5.2.1). Hence we recover the well-known fact:

(1) LEMMA. $0 \to F \xrightarrow{i_J} J \xrightarrow{\rho_J} A \to 0$ *is an exact sequence of algebraic groups.*

PROOF. To show that i_F and ρ_J are algebraic morphisms merely embed the sequence in $P \xrightarrow{i_F} \tilde{J} \xrightarrow{\rho_J} A$ and apply $GAGA$.

5.5. Given $X_1, X_2 \subset J$ we define as usual: $X_1 + X_2 = \{x_1 + x_2 : x_i \in X_i\}$ and given subvarieties $X_i \subset \tilde{J}$ such that $X_i \cap J$ is dense in X_i we let $X_1 \oplus X_2$ be the Zariski-closure of $(X_1 \cap J) + (X_2 \cap J)$. Then $X_1 \oplus X_2$ is irreducible if X_1 and X_2 are and $\dim(X_1 \oplus X_2) \leqslant \dim X_1 + \dim X_2$. Moreover, if X_i^0 is a constructible Zariski-dense subset of $X_i \cap J$ then $X_1^0 + X_2^0$ is constructible and Zariski-dense in $X_1 \oplus X_2$.

6.

6.0. We now use the ideas in the preceding paragraphs to study the sets $W_r = u_r(\underline{C}^{(r)})$ in J. Let $\underline{C} = C - S$ and $\underline{C}' = C' - \pi^{-1}(S)$.

(1) LEMMA. *The map* $u \circ \pi$: $\underline{C}' \to J$ *extends to a complex analytic (hence algebraic) map* $\tilde{u} = C' \to \tilde{J}$.

PROOF. The diagram

$$\begin{array}{ccc} \underline{C} & \xrightarrow{u} & J \\ \big\downarrow{\scriptstyle \pi^{-1}} & & \big\downarrow{\scriptstyle \rho_J} \\ \underline{C}' & \xrightarrow{u'} & A \end{array}$$

commutes and u' extends to C'. Thus given $p \in \pi^{-1}(S)$ let U_α be a neighborhood of $u'(p)$ and s_α: $U_\alpha \to J$ be a flat section. Choose simple sections $\eta_i \in H$ and additional sections $\zeta_j \in H$ such that $\pi^* \zeta_j$ has zero residues and such that $\eta_1, \ldots, \eta_s, \zeta_1, \ldots, \zeta_d$ give a basis for $H/H' = H''$. Then given $x \in \rho_J^{-1}(U_\alpha)$ the homogeneous coordinates for $x - s_\alpha(\rho_J(x))$ in the fibre P are $[e^{x(\eta_1)}, \ldots, e^{x(\eta_s)}, x(\zeta_1), \ldots, x(\zeta_d), 1]$. Let U be a small disk about p in C' and let z be a coordinate on U. We choose a fixed point $p_0 \neq p$ in U and a fixed path l_0 in $\underline{C}'$ from the relevant base point b to p_0. Given $q \in U$ we agree to trace a path $l(q)$ from b to q by first traversing l_0 and then a path inside U. It is then clear that the functions

$$\exp\{u\pi(q)(\eta_i)\} = \exp\left\{\int_{l(q)} \pi^* \eta_i\right\} \quad \text{and} \quad u \circ \pi(q)(\zeta_j) = \int_{l(q)} \pi^* \zeta_j$$

are meromorphic in $z(q)$. This proves the lemma.

Let V_r be the Zariski-closure of W_r in $\tilde{J}$.

(2) Corollary. *$V_1 = u \circ \pi(C')$ is an algebraic subset of $\tilde{J}$ of pure dimension 1. If C is irreducible so is V_1.*

(3) Corollary.

$$V_r = \underbrace{V \oplus \cdots \oplus V}_{r\text{-times}}$$

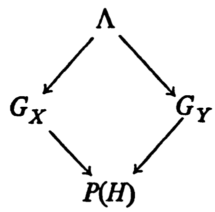
and V_r is of pure dimension r. W_r is a constructible subset of V_r and V_r is irreducible if C is.

Proof. The map u_r: $\underline{C}^{(r)} \to V_r \subset \tilde{J}$ is generically of rank r, so dim $V_r > r$. The rest is a trivial consequence of the lemma.

6.1. One easily verifies that the Zariski-closure of the graph of $u_r \circ \pi_r$: $(\underline{C}')^{(r)} \to V_r$ defines a birational correspondence r between $(C')^{(r)}$ and V_r. (π_r: $(C')^{(r)} \to C^{(r)}$ is the map induced by π.)

6.2. We agree to call $X \subset \tilde{J}$ *admissible* if $X \cap J$ is dense in X. Let X be an irreducible admissible hypersurface in $\tilde{J}$. In view of the triviality of $T^*(J)$ we may regard the annihilator $T_x(X)^{\perp}$ of the tangent space to X at any nonsingular point x as a line L_x in $T_0^*(J)$. More precisely, if $\tau_y(x) = x + y$ then $L_x = \tau_x^*(T_x(X)^{\perp})$. Thus, letting $P(H)$ denote the projective space of lines in H we have a correspondence $G_X^0 \subset (X_{\text{reg}} \cap J) \times P(H)$ and we let G_X be the Zariski-closure of G_X^0 in $X \times P(H)$. Of course, G_X is irreducible and G_X comes equipped with 2 projections, p_1: $G_X \to X$ and p_2: $G_X \to P(H)$, the first of which is birational.

(1) Proposition. *Let $X = x \oplus Y$. Then there exists a birational correspondence $\Lambda \subset G_X \times G_Y$ such that the following diagram is commutative:*



Proof. τ_x maps $Y_{\text{reg}} \cap J$ isomorphically onto $X_{\text{reg}} \cap J$ and $\tau_x^* T_{x+y}(X)^{\perp} = T_y(Y)^{\perp}$.

7.

7.0. In this section we collect some elementary facts about the branch locus of a morphism to projective space. Throughout this section morphism is to be taken in the algebraic sense. If X is an algebraic variety (over $\mathbf{C}$) we let $\mathbf{C}(X)$ denote the field of rational functions on X.

(1) Lemma. *Let $f: X \to Y$ be a proper surjective morphism from a variety X to a Zariski-open subset Y of affine space. If $f^{-1}(y)$ is a finite set for all y then* $\operatorname{card} f^{-1}(y) < [\mathbf{C}(X): \mathbf{C}(Y)]$ *for all $y \in Y$.*

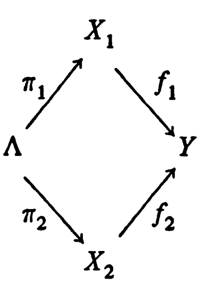
Proof. Let $N = [\mathbf{C}(X): \mathbf{C}(Y)]$. An elementary argument shows there exists a Zariski-open set $U \subset Y$ such that $\operatorname{card} f^{-1}(y) = N$ for all $y \in U$. Since X is pure dimensional as a complex analytic space and Y is locally irreducible, from the fact that $\dim f^{-1}(y) = 0$ for all $y \in Y$ we deduce that f is open in the complex topology. If $\operatorname{card} f^{-1}(y) = M > N$ then there exist disjoint open neighborhoods $U_1, \ldots, U_m$ about the points $x_i \in f^{-1}(y)$; and since f is open, there exists an open V such that $f(U_i) = V$ for each i (cutting down the U_i's if necessary). Then $\operatorname{card} f^{-1}(y) \geqslant M$ for all $y \in V$ but this is impossible since no such V can lie in the complement of a Zariski-open set on Y.

Let $f: X \to Y$ be a proper surjective morphism from a projective variety X to a projective space Y of the same dimension and $A_f = \{y: \operatorname{card} f^{-1}(y) \neq [\mathbf{C}(X): \mathbf{C}(Y)]\}$. We let $b(f)$ (the *branch locus* of f) be the union of the irreducible components X of the Zariski-closure of A_f such that $\operatorname{cod} X = 1$.

(2) Lemma. *Let $f: X \to Y$ be as above and suppose $b(f)$ is irreducible. Let X' be the normalization of X with projection $p: X' \to X$ and let $f' = f \circ p$. Then $b(f) = b(f')$.*

Proof. Since $f^{-1}(y)$ finite $\Rightarrow \operatorname{card}(f')^{-1}(y) > \operatorname{card} f^{-1}(y)$, clearly $b(f') \subset b(f)$. If $b(f') = \varnothing$ then $\bar{A}_{f'}$ has codimension 2 at least. But then $f': X' - (f')^{-1}(\bar{A}_{f'}) \to X - A_{f'}$ would be an N-sheeted cover of the simply connected space $Y - \bar{A}_{f'}$. This is not possible unless $N = 1$. But then f is birational and Y being normal we have $b(f) = \varnothing$. If $b(f') \neq \varnothing$ then certainly $b(f') = b(f)$.

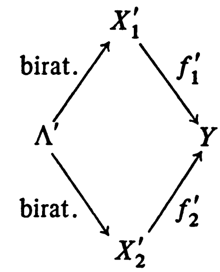
(3) Lemma. *Let $f_i: X_i \to Y$, $i = 1, 2$, be as above with X_i normal. Let Λ be projective variety with birational projections $\pi_i: \Lambda \to X$ such that the following diagram commutes:*



Then $b(f_1) = b(f_2)$.

Proof. There exists a closed subset A_1 of X_1 of codimension 2 such that $\pi_1|\Lambda - f_1^{-1}(A_1) \to X_1 - A_1$ is an isomorphism. Thus $b(f_1 \circ \pi_1) = b(f_1)$. Similarly $b(f_2 \circ \pi_2) = b(f_2)$. Hence $b(f_1) = b(f_2)$.

7.1. Remark. If we are given f_i: $X_i \to Y$ and a birational correspondence Λ as above then we also have a diagram:



with X_i' the normalization of X_i and Λ' the normalization of Λ.

7.2. From now on we assume C is *irreducible*. The *canonical mapping* ψ: $C \to P(H^*)$ associates to a point $p \in C$ the hyperplane $\{w \in H: w(p) = 0\}$ (we identify hyperplanes in H with lines in H^*). The map ψ is defined at p provided that there exists $w \in H$ such that $w(p) \neq 0$. In analogy with the nonsingular case, we call C *hyperelliptic* if ψ is not an embedding. Letting m_x be the sheaf germs of holomorphic functions on C which vanish at x we verify that C is nonhyperelliptic if and only if the following three sequences are exact:

(1) $0 \to H^0(C; m_x \otimes \Omega) \to H^0(C; \Omega) \to \mathbf{C} \to 0$,

(2) $0 \to H^0(C; m_x m_y \otimes \Omega) \to H^0(C; m_x\Omega) \to \mathbf{C} \to 0$ for all $x, y \in C$, $x \neq y$,

(3) $0 \to H^0(C; m_x^2\Omega) \to H^0(C; m_x\Omega) \to T_x^* \to 0$ for all $x \in C$. ($T_x^* \cong H^0(C; m_x/m_x^2)$ is the Zariski cotangent space of C.)

The three conditions correspond respectively to the geometric conditions:

(1)′ ψ is defined at all p.

(2)′ ψ separates points.

(3)′ ψ^*: $T^*_{\psi(p)} \to T_x^*$ is surjective.

Certainly every plane curve of degree $d \geq 4$ is nonhyperelliptic because the canonical bundle of $P(\mathbf{C}^2)$ is just h^{-3} where h is the hyperplane section bundle of $P(\mathbf{C}^2)$ and the line bundle $[C]$ is h^d. Thus $K = h^{d-3}|C$ and since sections of h^{d-3} embed $P(\mathbf{C}^2)$, *a fortiori* they embed C.

Remarks. (a) Condition (1) is satisfied if $g \geq 1$. Hereafter we shall assume that this is the case.

(b) If condition (2) is satisfied then $g \geq 3$ and condition (3) holds at each nonsingular point x.

(c) If $g \geqslant 2$ and $x \neq y$ then $\psi(x) \neq \psi(y)$; if not both x and y are nonsingular.

(d) Condition (2) fails if C carries a line bundle ζ of degree 2 such that $h^0(\zeta) = 2$.

(e) If (2) fails then ψ is generically 2 to 1.

(f) If x is an ordinary n-fold singularity with $n \geqslant 3$ then (3) holds at x.

7.3. Given any Cartier divisor d on C let $h^0(d) = \dim H^0(C, \mathcal{O}(d))$ and then define $W_r^2 = \{u_r(d)\colon d \in \underline{C}^{(r)} \text{ and } h^0(d) \geqslant 2\}$. In the case when C is nonsingular the following important facts about the sets W_r^2 were observed by Riemann and given proofs by Martens [5] and more recently Saint-Donat [9].

(1) Theorem. *For all* r, $2 \leqslant r \leqslant g - 1$, $\dim W_r^2 \leqslant r - 2$. *Moreover, if for some* r, $\dim W_r^2 = r - 2$, *then* $W_2^2 \neq \varnothing$. (*In particular C is hyperelliptic.*)

Saint-Donat's proof may be modified to apply in the singular case. Of course it then becomes necessary to define: $\dim W_r^2 = \dim$(Zariski-closure of W_r^2 in J). As a consequence we have the following:

(2) Theorem. *Let* Σ_r *be an algebraic subset of* $\underline{C}^{(r)}$ *such that* $d \in \Sigma_r \Rightarrow h^0(d) \geqslant 2$. *If* $\dim \Sigma_r \geqslant r - 1$, $W_2^2 \neq \varnothing$.

Proof. Let Σ_r be as above but with r as small as possible. Then there exists $d \in \Sigma_r$ such that $h^0(d) = 2$. Otherwise for all $d \in \Sigma_r$ we would have $h^0(d) \geqslant 3$. Then if we decompose d into a sum $p + d'$ with $p \in \underline{C}$ and $d' \in \underline{C}^{(r-1)}$ we have $h^0(d') \geqslant 2$. Taking the set of all such d' we arrive at an algebraic set $\Sigma_{r-1} \subset \underline{C}^{(r-1)}$ such that $d \in \Sigma_{r-1}$, $h^0(d) \geqslant 2$, and $\dim \Sigma_{r-1} \geqslant r - 2$. Thus we may assume $h^0(d) = 2$ for generic $d \in \Sigma_r$. Since the fibre $u_r^{-1}(u_r(d))$ of the map $u_r\colon \Sigma_r \to W_r^2$ is parameterized by a subspace of the space of divisors linearly equivalent to d we have $\dim u_r^{-1}(u_r(d)) \leqslant 1$ for almost all d. Thus $\dim \overline{u_r(\Sigma_r)} \geqslant r - 2$. The result now follows from Theorem 1.

7.4. From now on when given a section λ of a line bundle L over C we shall let (λ) be the divisor on C' of the section $\lambda \circ \pi$ of the pullback $L' = L \circ \pi$. As usual $d \leqslant d' \Leftrightarrow$ the divisor d is contained in d' (both d and d' effective). Then the *canonical correspondence* Λ is the closure of $\Lambda^0 = \{(d, [w]) \in (\underline{C}')^{(g-1)} \times P(H)$ such that $(w) \geqslant d$ and $h^0(d) = 1\}$. We let $\Lambda^1 = \{(d, [w]) \in (C')^{(g-1)} \times P(H)$ such that $(w) \geqslant d\}$. The natural projections from Λ (resp. Λ^1) to $(C')^{(g-1)}$ and $P(H)$ we denote by π_1 and π_2 (resp. π_1^1 and π_2^1). We shall show that if $W_2^2 = \varnothing$ then $\Lambda = \Lambda^1$ by first showing that $(\pi_2^1)^{-1}(w) \subset \Lambda$ for some w and then "specializing" to conclude that $\Lambda^1 \subset \Lambda$. The result is important because the branch locus of π_2 will be seen to have geometric significance and this result allows us to compute the branch locus. The following lemma was proved by Andreotti [1] in the nonsingular case though by a rather different argument. In the course of our argument when given a divisor $d \in (\underline{C}')^{(r)}$ we let $h^0(d) = \dim H^0(C, \mathcal{O}(\pi_* d))$. (Thus we compute the dimension of d by regarding d as a divisor on C.)

(1) LEMMA. *If* $W_2^2 = \varnothing$ *then* $\exists w \in H$ *such that* $|(w)| \subset \underline{C}'$ *and* $d \in (C')^{(g-1)}$ *and* $d < (w) \Rightarrow h^0(d) = 1$.

PROOF. Supposing the lemma false, let r be the least integer such that $|(w)| \subseteq \underline{C}' \Rightarrow \exists d \in (C')^{(r)} \ni d \leqslant w$ and $h^0(d) \geqslant 2$. Now form the correspondence $\Lambda^2 \subset (C')^{(r)} \times P(H)$ by taking the closure of the set of pairs $(d, [w])$ with the properties above. Then $\dim \Lambda^2 = g - 1$. Moreover, if p_1 and p_2 are the canonical projections then for generic d, $\dim p_1^{-1}(d) = h^0(\Omega(-d)) - 1 \geqslant g - r$. It follows that $\dim p_1(\Lambda^2) \geqslant r - 1$ unless $h^0(\Omega(-d)) \geqslant g - r + 1$ for all $d \in p_1(\Lambda^2) = \Sigma_r$. In the latter case r would not be minimal (cf. proof of Theorem 2). Hence $\dim \Sigma_r = r - 1$, so by Theorem 2, $W_2^2 \neq \varnothing$.

(2) LEMMA. *If* $W_2^2 = \varnothing$ *then* $\Lambda = \Lambda^1$.

PROOF. By Lemma 1, $\exists [w] \in P(H)$ such that $|(w)| \subset \underline{C}'$ and $(d, [w]) \in \Lambda^1 \Rightarrow (d, [w]) \in \Lambda$. Moreover, the set U of such $[w]$ is clearly Zariski open in $P(H)$ and $[w] \in U \Rightarrow (\pi_2^1)^{-1}[w] \subset \Lambda$. Given an arbitrary $[w_0] \in P(H)$ we choose $[w^n] \in U$ such that $[w^n]$ converges to $[w_0]$ in the strong topology. Next we show that given $d \in (\pi_2^1)^{-1}[w_0]$, $\exists d^n \in \pi_2^{-1}[w^n]$ such that $d^n \to d$ in $(C')^{(g-1)}$.

It suffices to show that if $d = rp \leqslant (w_0)$, there exist $d^n \leqslant (w^n)$ such that $d^n \to rp$. To see that this is the case let Δ be a small disk centered at p with coordinate t and e be a local nonvanishing section of $K' = K \circ \pi$. Choose a basis $w_0, \ldots, w_{g-2}$ for H and suppose $w_i \circ \pi = f_i \cdot e$ in Δ. Then $[w^n] = [\Sigma x_i^n w_i]$ (we require that $x_0^n = 1$). If P_j: $U^n \to U$ is the jth projection we construct coordinates s_j on $U^{(r)}$ via: $s_j = \Sigma_{k=1}^r (t \circ p_k)^j$, $j = 1, \ldots, r$. Now the divisor d^n of $w^n \circ \pi | U$ has coordinates:

$$s_j(d^n) = \int_{\partial\Delta} t^j \left(\sum_{k=0}^{g-2} x_k^n f_k'(t) \right) \left(\sum_{k=0}^{g-2} x_k^n f_k(t) \right)^{-1} dt$$

and since $\lim_{n\to\infty} x_k^n = 0$ if $k \geqslant 1$ we conclude that $\lim_{n\to\infty} s_j(d^n) = s_j(d)$. We have shown that Λ is dense in Λ^1 and since Λ is closed $\Lambda = \Lambda^1$.

By the Riemann-Roch theorem we have $\deg(w) = 2g - 2$ for the canonical divisor. Hence card $\pi_2^{-1}[w] < \binom{2g-2}{g-1}$. In the next section we shall show among other things that generically (w) contains $2g - 2$ distinct points, hence generically card $\pi_2^{-1}[w] = \binom{2g-2}{g-1}$.

(3) COROLLARY. *If* $W_2^2 = \varnothing$ *then the branch locus* $b(\pi_2)$ *is the set* $B = \{[w]$: $(w) = 2p + \ldots$ *for some* $p \in C'\}$.

PROOF. If $(w) = 2p_1 + \cdots + p_{2g-2}$ where $p_i \neq p_j$ if $i \neq j$, then since

$\Lambda^1 = \Lambda$, card $\pi_2^{-1}[w] = \binom{2g-2}{g-1} = N$. If $(w) = 2p_1 + \cdots + p_{2g-3}$ then

$$\text{card}\, \pi_2^{-1}[w] \leq \binom{2g-2}{g-1} - \binom{2g-4}{g-3} < N.$$

The proof is complete upon noting that B is of pure codimension 1 in $P(H)$.

As a result of this corollary we see that B contains the set of hyperplanes $[w]$ in $P(H^*)$ which are tangent to the canonical image $X = \psi(C)$.

7.5. In general, given a variety X in a projective space P of dimension N we have both a *dual correspondence* in $X \times P^*$

$$\mathfrak{D}(X) = \text{Zariski-closure}\ \{(x, h): x \text{ is n.s. on } X \text{ and } h \supset T_x(X)\}$$

and a *dual variety* $X^* = \pi_2\mathfrak{D}(X) \subset P^*$.

(1) Lemma. (a) *X irreducible $\Rightarrow \mathfrak{D}(X)$ and X^* irreducible.*

(b) $\dim \mathfrak{D}(X) = N - 1$.

(c) $X^{**} = X$.

Proof. (a) Let $A = \{(x, h): x \text{ is n.s. on } X \text{ and } h \supset T_x(X)\}$ and let $m = \dim X$. Then $\pi_1: A \to X_{\text{reg}}$ is proper and $\pi_1^{-1}(x)$ is a projective space of dimension $N - m - 1$. Hence X irreducible $\Rightarrow A$ irreducible $\Rightarrow \mathfrak{D}(X) = \bar{A}$ irreducible $\Rightarrow X^*$ irreducible.

(b) Since $\dim \pi_1^{-1}(x)$ is $N - m - 1$ (generically) $\dim \mathfrak{D}(X) = \dim X + N - m - 1 = N - 1$.

(c) It suffices to prove the result for X irreducible. Then since $\mathfrak{D}(X^*)$ is irreducible and of dimension $N - 1$ it suffices to show $(x, h) \in \mathfrak{D}(X) \Rightarrow (h, x) \in \mathfrak{D}(X^*)$. Let $z = (x, h) \in \mathfrak{D}(X)$ be a n.s. point on $\mathfrak{D}(X)$ such that

(i) x is n.s. on X and h is n.s. on X^* and

(ii) $(\pi_2)_* T_z\mathfrak{D}(X) = T_h(X^*)$.

Given a tangent vector $v \in T_h(X^*)$ choose a (holomorphic) curve $\gamma(t)$ in $\mathfrak{D}(X)$ such that $\gamma(0) = (x, h)$ and $(\pi_2)_*\gamma'(0) = v$ and let $\sigma(t) = \pi_1\gamma(t)$, $\tau(t) = \pi_2(\gamma(t))$. Choose affine coordinates near x (and dual coordinates near h) so that

$$\sigma(t) = [1, \sigma_1(t), \ldots, \sigma_N(t)], \quad \tau(t) = [\tau_0(t), 1, \ldots, \tau_N(t)].$$

Then $\gamma(t) \in \mathfrak{D}(X) \Rightarrow \Sigma\sigma_i\gamma_i = 0$ and $\Sigma\sigma_i'\gamma_i = 0$. Therefore $\Sigma\sigma_i\gamma_i' = 0$ or $\Sigma x_i v_i = 0$ (letting $t = 0$). Since this holds for any tangent vector v we have $(h, x) \in \mathfrak{D}(X^*)$. The set of such $z = (x, h)$ is dense in $\mathfrak{D}(X)$ and this proves the lemma.

(2) Corollary. *If X is the canonical image of a curve C for which $W_2^2 = \varnothing$ then the generic hyperplane $[w] \in P(H)$ is not tangent to X and does not pass through $\psi(S)$ hence (w) contains $2g - 2$ distinct points.*

Proof. $\dim X^* = g - 2$ and ψ is bijective as is $\pi|\underline{C}'$.

(3) LEMMA. *If* $W_2^2 \neq \varnothing$ *then* ψ *is generically* $(2-1)$ *and* $\Lambda = \{(d, [w]): (\psi \circ \pi)_* d = [w] \cdot X\}$. (*Here* $[w] \cdot X$ *is the divisor of intersection of the hyperplane* $[w]$ *in* $P(H^*)$ *with the curve* X.)

PROOF. Let L be the (unique) line bundle of degree 2 on C such that $h^0(L) = 2$. If $[p_1 + p_2] = L$ then $\psi(p_1) = \psi(p_2)$ by the R.R. theorem, i.e. any section $w \in H$ which vanishes at p_1 vanishes at p_2 as well. Thus $w \in H \Rightarrow (w) = \Sigma_{i=1}^{g-1} d_i$ where $\pi_* d_i$ is the divisor of a section of L. Clearly if $d \leqslant (w)$ and $\deg d = g - 1$ then $h^0(d) = 1 \Leftrightarrow d$ contains exactly one point from each of the divisors $d_i \Leftrightarrow (\psi \circ \pi)_* d = [w] \cdot X$. The lemma now follows from an argument similar to that given in Lemma 7.4.2.

Let $B = \varnothing$ if $W_2^2 = \varnothing$ and $B = \{q\colon q \in X - \psi(S)$ and $\psi^{-1}(q)$ is a singleton$\}$ if $W_2^2 \neq \varnothing$. Given $q \in P(H^*)$ let $\operatorname{star}(q) = \{[w]\colon [w] \in q\}$ and define $Y_B = \bigcup \{\operatorname{star}(q)\colon q \in B\}$ and $Y_S = \{\operatorname{star}(q)\colon q \in \psi(S)\}$.

(4) LEMMA. $X^* \cup Y_B \subset b(\pi_2) \subset X^* \cup Y_B \cup Y_S$.

PROOF. If $W_2^2 = \varnothing$ and $(W) = 2p_1 + \cdots + p_{2g-3}$ then either $p_1 \in S^1 - \pi^{-1}(S)$ in which case $[w]$ is tangent to X at $(\psi \circ \pi)(p_1)$ and $[w] \in X^*$, or $p_1 \in S'$ in which case $[w] \in \operatorname{star}(\psi \circ \pi(p_1))$. Thus $b(\pi_2) \subset X^* \cup Y_S$ as required.

If $W_2^2 \neq \varnothing$ again let L be the line bundle on C such that $\deg(L) = h^0(L) = 2$. If $[w] \notin X^* \cup Y_B \cup Y_S$ then $(w) = \Sigma_{i=1}^{g-1} p_i + p_i'$ where $[\pi(p_i) + \pi(p_i')] = L$, $p_i \neq p_j'$ and $p_i \neq p_j$ if $i \neq j$, and $p_i \neq p_i'$. We have $\psi \circ \pi(p_i) = \psi \circ \pi(p_i')$ while $\psi \circ \pi(p_i) \neq \psi \circ \pi(p_j)$ if $i \neq j$. It follows that $\operatorname{card}(\pi_2^{-1}[w]) = 2^{g-1}$ (apply Lemma (3)). On the other hand if $[w] \in \operatorname{star}(b)$ for some $b \in B$ then $\operatorname{card}(\pi_2^{-1}[w]) \leqslant 2^{g-2}$ while if $[w]$ is tangent to X at a nonsingular point $\operatorname{card} \pi_2^{-1}[w] \leqslant 2^{g-3}3$. This proves the lemma.

In the sequel we shall need to know that after normalization of Λ to obtain Λ' the new branch locus still contains the component X^*. This is a consequence of the following:

(5) THEOREM. *Let* $[w]$ *be simply tangent to* X *at exactly one point and suppose* $[w] \notin Y_B \cup Y_S$. *Then* Λ *is analytically irreducible at each point in* $\pi_2^{-1}[w]$. (*If* $W_2^2 \neq \varnothing$ *then* Λ *is in fact nonsingular at each point in* $\pi_2^{-1}[w]$.)

PROOF. If $W_2^2 = \varnothing$ then $(w) = 2p_1 + \cdots + p_{2g-3}$ where $p_i \neq p_j$ if $i \neq j$. Choose $(d_0, [w_0]) \in \pi_2^{-1}([w_0])$ and select a basis $w_0, \ldots, w_{g-2}$ for H. If $[w] = [\Sigma x_i w_i]$ then the ratios $\psi_i = x_i / x_0$ $(i \geqslant 1)$ form a set of analytic coordinates on a neighborhood of $[w_0]$. Hereafter, given w in this neighborhood we shall suppose $x_0 = 1$. Choose a coordinate disk $U_i \subset C'$ centered at p_i so that the disks are disjoint and the bundle $\pi^* K$ has a nonvanishing section e_i over U_i. Denote the analytic coordinate in U_i by t_i and suppose that $t_i(p_i) = 0$. Then $w_j = f_{ji} e_i$, where f_{ji} is holomorphic.

There are now two cases to consider according as d_0 has multiplicity 1 or 2 at p_i. It suffices to consider $d_0 = 2p_1 + p_2 + \cdots + p_{g-2}$ and $d_0 = p_1 + p_2 + \cdots + p_{g-1}$.

In the first case $U = U_1^{(2)} \times U_2 \times \cdots \times U_{g-2}$ is a coordinate neighborhood of d_0 in $(C')^{(g-1)}$. If t_0' and t_1' denote the compositions with t_1 of the projections from $U_1 \times U_1$ to U_1 then $s_1' = t_0' + t_1'$ and $s_0' = (t_0')^2 + (t_1')^2$ are coordinates on $U_1^{(2)}$ and these composed with the projections from U to $U_1^{(2)}$ give coordinates s_0 and s_1 on U. Finally, for $k \geqslant 2$ let $s_k = t_k \circ \pi^k$ where π^k: $U \to U_k$ is the projection. Then $s_0, \ldots, s_{g-2}$ form a complete set of coordinates on U.

If $w = w_0 + \Sigma x_i w_i$ we let $\lambda_j(x, t) = f_{0j} + \Sigma x_j f_{ij}(t)$ and $g_j(x, t) = D_t\lambda_j(x, t)/\lambda_j(x, t)$. Then the divisor of w restricted to U_j has degree $(2\pi i)^{-1}\int_{\partial U_j} g_j(x, t)\, dt$. Since these functions are integer valued and analytic in x they must be constant on some neighborhood V of $x = 0$. It follows that if $(d, [w]) \in \Lambda \cap (U \times V)$ then $s_0(d) = \int_{\partial U_1} t^2 g_1(x, t)\, dt$ and $s_k(d) = \int_{\partial U_k} t g_k(x, t)\, dt$ for $k \geqslant 1$. Cutting down V if necessary will insure that the values of the functions s_i defined by the above equations are indeed coordinates of a divisor in U. This shows that the projection π_2: $\Lambda \cap (U \times V) \to V$ is an isomorphism, so Λ is irreducible at $(d_0, [w_0])$.

If $d_0 = p_1 + \cdots + p_{g-1}$ with $U = U_1 \times \cdots \times U_{g-1}$ and V as above then $(d, [w]) \in \Lambda \cap (U \times V) \to s_k(d) = \int_{\partial U_k} t g_k(x, t)\, dt$ if $k \geqslant 2$ while $s_1(d)$ is a root of the polynomial $Q(s) = s^2 - \alpha s + \beta$ where $\alpha = \int_{\partial U_1} t g_1(x, t)\, dt$ and $\beta = (\alpha^2 - \int_{\partial U_1} t^2 g_1(x, t)\, dt)/2$. The discriminant of Q is $\Delta = \alpha^2 - 4\beta = 2\int_{\partial U_1} t^2 g_1(x, t) - \alpha^2$. Both α and β vanish at $x = 0$ hence so does Δ. Moreover,

$$D_{x_k}\Delta(0) = 2\int_{\partial U_1} t^2 (D_{x_k} D_t \lambda_j(0, t) - D_t\lambda_j(0, t) D_{x_k}\lambda_j(0, t))\lambda_j^{-2}(0, t)\, dt$$

$$= 2\int_{\partial U_1} t^2 (D_t f_{k1}(t) f_{01}(t) - D_t f_{01}(t) f_{k1}(t)) f_{01}^{-2}\, dt.$$

By assumption f_{01} has a zero of order exactly 2 at 0. Thus $t^2 f_{01}^{-1}$ is holomorphic and the 1st term integrates to zero. However, $t^2 D_t f_{01}/f_{01}^2$ has a pole of order 1 at 0 and since at least one section w_i does not vanish at p_1 at least one function f_{k1} is not zero at 0. Accordingly $D_{x_k}\Delta(0) \neq 0$ for some k. It follows that Q is irreducible. Thus $\Lambda \cap (U \times V)$ is an irreducible quadratic variety over V and this completes the proof when $W_2^2 = \varnothing$. The other case is omitted as the proof is similar.

8.

8.0. In this section we prove the main theorems of this paper.

The first formulation is:

(1) THEOREM. *Let C be an irreducible curve of genus $\geqslant 2$. Let $\theta = V_{g-1}$ and*

let G'_θ be the normalization of the Gauss correspondence G_θ. Then the branch locus of π_G: $G'_\theta \to P(H)$ equals the branch locus of π'_2: $\Lambda' \to P(H)$.

PROOF. The closed graph of $u_{g-1} \circ \pi_{g-1}$: $(\underline{C}')^{(g-1)} \to \theta$ is a birational correspondence between $(C')^{(g-1)}$ and θ, hence induces a correspondence T between Λ and G_θ. First we must show that the diagram

$$\begin{array}{ccc} & T & \\ {}^{P_1}\swarrow & & \searrow \\ \Lambda & & G_\theta \\ \searrow & & \swarrow \\ & P(H) & \end{array}$$

commutes. Let $U_1 = \{d \in (C')^{(g-1)}: h^0(d)\ (= h^0(\Omega(-d))) = 1\}$ and $U_2 = \{x \in \theta: x$ is n.s. on $\theta\}$. Let $f = u_{g-1} \circ \pi_{g-1}$. Then $f(U_1)$ is dense in θ hence in U_2 and $U' = f^{-1}(U_2) \cap U_1$ is dense and open in $(\underline{C}')^{(g-1)}$. Moreover, f has rank $g-1$ at each $d \in U'$ so $f_* T_d(U') = T_{f(d)}(\theta)$. But by 4.5.1, $(f_* T_d(U')) = \{w: w > d\ w \in H\}$ and this is precisely the line in $P(H)$ associated to d by Λ.

Since the diagram commutes on a dense set, it commutes. It follows that

$$\begin{array}{ccc} & T' & \\ \swarrow & & \searrow \\ \Lambda' & & G'_\theta \\ {}^{\pi'_2}\searrow & & \swarrow^{\pi_G} \\ & P(H) & \end{array}$$

commutes and therefore $b(\pi'_2) = b(\pi_G)$. This completes the proof.

(2) THEOREM. *Let C_1 and C_2 be irreducible curves. Let f: $F_1 \to F_2$, j: $J_1 \to J_2$, a: $A_1 \to A_2$ be analytic isomorphisms such that*:

(a) *The diagram*

$$(*) \qquad \begin{array}{ccccccccc} 0 & \to & F & \to & J & \to & A & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & F_2 & \to & J_2 & \to & A_2 & \to & 0 \end{array}$$

commutes.

(b) *f respects the canonical splitting of the Lie algebras of F_1 and F_2. (Alternatively f is an algebraic isomorphism.)*

(c) *$j(\theta_1') = \theta_2' + x$ for some x in J_2. (Here θ_i' is the Zariski-closure of W_{g-1} in J_i rather than in $\tilde{J}_i$.)*

Then j_*: $T_0(J_1) \to T_0(J_2)$ induces a projective linear isomorphism $P(H_1^*) \to P(H_2^*)$ which carries X_1 into X_2.

PROOF. Condition (b) insures that f is an algebraic isomorphism. Since A_1 is compact and projective algebraic, the map a is also an algebraic isomorphism. It follows that j is algebraic as well. Choosing compactifications $\tilde{J}_i$ as in (5.0) we may take the Zariski-closure of the graph of j in $\tilde{J}_1 \times \tilde{J}_2$ and it is immediate that this induces a birational correspondence between θ_1 and $\theta_2 \oplus x$ hence also between G_{θ_1} and $G_{\theta_1 \oplus x}$. Moreover, since j is a homomorphism the following diagram commutes:

$$\begin{array}{ccccc} G_{\theta_1} & \longleftarrow & T & \longrightarrow & G_{\theta_2 \oplus x} \\ \downarrow{\scriptstyle \pi_1} & & & & \downarrow{\scriptstyle \pi_2} \\ P(H_1) & & \xrightarrow{\quad (j^*)^{-1} \quad} & & P(H_2) \end{array}$$

It follows from 7.0.3 that $(j^*)^{-1}X_1^* \subset b(\pi_2)$ and from 6.2.1 that modulo linear components $b(\pi_2) = X_2^*$. Since neither X_1^* nor X_2^* is a hyperplane (or else $X_i = X_i^{**}$ would be a point) it follows that $(j^*)^{-1}(X_1^*) = X_2^*$ as required.

If in addition to the assumptions of the theorem it happens that neither X_1 nor X_2 is hyperelliptic, then, of course, the curves C_1 and C_2 are isomorphic as a consequence of the theorem. However, in the classical case this condition on C_1 and C_2 is not necessary as it is possible to reconstruct the curve C from a knowledge of the branch points of the map ψ: $C \to X$ together with the fact that X is a rational curve. In the present case, unfortunately, the map ψ can fail to be an embedding in several ways and it is not clear to the author that the branch locus of G_θ contains enough information to recover C.

REFERENCES

1. A. Andreotti, *On a theorem of Torelli*, Amer. J. Math. **80** (1958), 801–827.
2. R. C. Gunning, *Lectures on Riemann surfaces, Jacobi varieties*, Princeton Univ. Press, Princeton, N. J., 1972.
3. K. Kodaira, *On compact complex analytic surfaces*. I, Ann. of Math. (2) **71** (1962), 111–152.
4. ———, *On Kähler varieties of restricted types*, Ann. of Math. **60** (1954), 28–48.
5. H. H. Martens, *On the varieties of special divisors on a curve*, J. Reine Angew. Math. **227** (1967), 111–120.
6. A. Mayer, *Special divisors and the Jacobian variety*, Math. Ann. **153** (1964), 163–167.
7. R. Narasimhan, *Introduction to the theory of analytic spaces*, Springer-Verlag, Berlin, 1966, 132 pp.
8. M. Rosenlicht, *Generalized Jacobian varieties*, Ann. of Math. (2) **59** (1954), 505–530.

9. B. Saint-Donat, *On Petri's analysis of the linear system of quadrics through a canonical curve*, Math. Ann. **206** (1973), 156–175.

10. J.-P. Serre, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier (Grenoble) **6** (1955–1956), 1–42.

11. ______, *Groupes algébriques et corps de classes*, Hermann, Paris, 1959.

12. F. Severi, *Funzioni quasi-abeliane*, Vatican, 1947.

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